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On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space

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Abstract
In this paper, we give strong convergence theorems for the modified S-iteration process of asymptotically quasi-nonexpansive mappings on a CAT(0) space which extend and improve many results in the literature.

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Keywords: CAT(0) space; asymptotically quasi-nonexpansive mapping; strong convergence; iterative process; fixed point

1 Introduction

A metric space X is a CAT(0) *space* if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The initials of the term 'CAT' are in honor of E. Cartan, A. D. Alexanderov and V. A. Toponogov. A CAT(0) space is a generalization of the Hadamard manifold, which is a simply connected, complete Riemannian manifold such that the sectional curvature is non-positive. In fact, it is very well known that any complete simply connected Riemannian manifold with non-positive sectional curvature is a CAT(0) space. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space (see [1]). Other examples include Pre-Hilbert spaces, R-trees (see [2]) and Euclidean buildings (see [3]). A CAT(0) space plays a fundamental role in various areas of mathematics (see Bridson and Haefliger [2], Burago, Burago and Ivanov [4], Gromov [5]). Moreover, there are applications in biology and computer science as well [6, 7].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [8, 9]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has been rapidly developed and many papers have appeared (see, *e.g.*, [8–14]). It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see [2, p.165]). Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)x_n + a_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}$ is a sequence in $(0,1)$.

Further, the Ishikawa iteration process is defined as the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)x_n + a_n T y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0,1)$. This iteration process reduces to the Mann iteration process when $b_n = 0$ for all $n \in \mathbb{N}$.

Agarwal, O'Regan and Sahu [15] introduced the S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)T x_n + a_n T y_n, \\ y_n = (1-b_n)x_n + b_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences in $(0,1)$. Note that (1.3) is independent of (1.2) (and hence of (1.1)). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these (see [15, Proposition 3.1]).

Schu [16], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)x_n + a_n T^n x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{a_n\}$ is a sequence in $(0,1)$.

Tan and Xu [17], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. This iteration process reduces to the modified Mann iteration process when $b_n = 0$ for all $n \in \mathbb{N}$.

Recently, Agarwal, O'Regan and Sahu [15] introduced the modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)T^n x_n + a_n T^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are in $(0,1)$. Note that (1.6) is independent of (1.5) (and hence of (1.4)). Also, (1.6) reduces to (1.3) when $n = 1$.

We now modify (1.6) in a CAT(0) space as follows.

Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - a_n)T^n x_n \oplus a_n T^n y_n, \\ y_n = (1 - b_n)x_n \oplus b_n T^n x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where and throughout the paper $\{a_n\}$, $\{b_n\}$ are the sequences such that $0 \leq a_n, b_n \leq 1$ for all $n \in \mathbb{N}$.

In this paper, we study the modified S-iteration process for asymptotically quasi-nonexpansive mappings on the CAT(0) space and generalize some results of Khan and Abbas [14] which studied the S-iteration process in a CAT(0) space for nonexpansive mappings. This paper contains three sections. In Section 2, we first collect some known preliminaries and lemmas that will be used in the proofs of our main theorems. We give the main results related to the strong convergence theorems of the modified S-iteration process in a CAT(0) space in Section 3. Under some suitable condition, we obtain the main theorems which state that $\{x_n\}$ converges strongly to a fixed point of T. Our results can be applied to an S-iteration process since the modified S-iteration process reduces to the S-iteration process when $n = 1$.

2 Preliminaries and lemmas

Let us recall some definitions and known results in the existing literature on this concept.

Let (X, d) be a metric space and K its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

The concept of quasi-nonexpansiveness was introduced by Diaz and Metcalf [18] in 1967, the concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [19] in 1972. The iterative approximation problems for asymptotically quasi-nonexpansive mapping were studied by Liu [20], Fukhar-ud-din *et al.* [21], Khan *et al.* [22] and Beg *et al.* [23] in a Banach space and a CAT(0) space.

Definition 1 Let (X, d) be a metric space and K be its nonempty subset. Then $T : K \to K$ is said to be

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$,
(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$,
(3) quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in K$, $p \in F(T)$,
(4) asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for all $x \in K$, $p \in F(T)$,
(5) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Remark 1 From Definition 1, it is clear that the class of quasi-nonexpansive mappings and asymptotically nonexpansive mappings includes nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive mappings is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings. The reverse of these implications may not be true.

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x,y \in X$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) = \triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$$

for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [2]).

A geodesic metric space is said to be a CAT(0) space [2] if all geodesic triangles of an appropriate size satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \le d_{\mathbb{R}^2}(\overline{x},\overline{y}).$$

A complete CAT(0) space is often called *Hadamard space* (see [24]).

Finally, we observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

The equality holds for the Euclidean metric. In fact (see [2, p.163]), a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1) (which is known as the *CN* inequality of Bruhat and Tits [25]).

Let $x,y \in X$, by [12, Lemma 2.1(iv)] for each $t \in [0,1]$, then there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{2.2}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.2). By using this notation, Dhompongsa and Panyanak [12] obtained the following lemma which will be used frequently in the proof of our main results.

Lemma 1 *Let X be a* CAT(0) *space. Then*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z)$$

for all $t \in [0,1]$ and $x, y, z \in X$.

The following lemma can be found in [26].

Lemma 2 *Let $\{a_n\}$ and $\{u_n\}$ be two sequences of positive real numbers satisfying*

$$a_{n+1} \leq (1+u_n)a_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} u_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3 Main results

In this section we prove the strong convergence theorems of the modified S-iteration process in a CAT(0) space.

Theorem 1 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, $T : K \to K$ be asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{u_n\}$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} u_n < \infty$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.7). *If*

$$\liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0 \quad \textit{or} \quad \limsup_{n\to\infty} d\big(x_n, F(T)\big) = 0,$$

where $d(x, F(T)) = \inf_{z\in F(T)} d(x,z)$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Let $p \in F(T)$. Since T is an asymptotically quasi-nonexpansive mapping, there exists a sequence $\{u_n\} \in [0,\infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that

$$d\big(T^n x, p\big) \leq (1+u_n)d(x,p)$$

for all $x \in K$ and $p \in F(T)$. By combining this inequality and Lemma 1, we get

$$\begin{aligned} d(y_n,p) &= d\big((1-b_n)x_n \oplus b_n T^n x_n, p\big) \\ &\leq (1-b_n)d(x_n,p) + b_n d\big(T^n x_n, p\big) \\ &\leq (1-b_n)d(x_n,p) + b_n(1+u_n)d(x_n,p) \\ &= (1+b_n u_n)d(x_n,p). \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} d(x_{n+1},p) &= d\big((1-a_n)T^n x_n \oplus a_n T^n y_n, p\big) \\ &\leq (1-a_n)d\big(T^n x_n, p\big) + a_n d\big(T^n y_n, p\big) \\ &\leq (1-a_n)(1+u_n)d(x_n,p) + a_n(1+u_n)d(y_n,p) \end{aligned}$$

$$
\begin{aligned}
&\le (1-a_n)(1+u_n)d(x_n,p) + a_n(1+u_n)(1+b_nu_n)d(x_n,p) \\
&\le (1-a_n)(1+u_n)d(x_n,p) + a_n(1+u_n)^2 d(x_n,p) \\
&= (1+u_n)(1-a_n+a_n+a_nu_n)d(x_n,p) \\
&\le (1+u_n)(1+u_n)d(x_n,p) \\
&= (1+u_n)^2 d(x_n,p).
\end{aligned}
\tag{3.2}
$$

When $x \geq 0$ and $1+x \leq e^x$, we have $(1+x)^2 \leq e^{2x}$. Thus,

$$
\begin{aligned}
d(x_{n+m},p) &\le (1+u_{n+m-1})^2 d(x_{n+m-1},p) \\
&\le e^{2u_{n+m-1}} d(x_{n+m-1},p) \\
&\le \cdots \\
&\le e^{2\sum_{k=n}^{n+m-1} u_k} d(x_n,p).
\end{aligned}
$$

Let $e^{2\sum_{k=n}^{n+m-1} u_k} = M$. Thus, there exits a constant $M > 0$ such that

$$d(x_{n+m},p) \le Md(x_n,p)$$

for all $n, m \in \mathbb{N}$ and $p \in F(T)$. By (3.2),

$$d(x_{n+1},p) \le (1+u_n)^2 d(x_n,p).$$

This gives

$$d\big(x_{n+1},F(T)\big) \le (1+u_n)^2 d\big(x_n,F(T)\big) = \big(1+2u_n+u_n^2\big)d\big(x_n,F(T)\big).$$

Since $\sum_{n=1}^{\infty} u_n < \infty$, we have $\sum_{n=1}^{\infty}(2u_n+u_n^2) < \infty$. Lemma 2 and $\liminf_{n\to\infty} d(x_n,F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n,F(T)) = 0$ gives that

$$\lim_{n\to\infty} d\big(x_n,F(T)\big) = 0. \tag{3.3}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Since $\lim_{n\to\infty} d(x_n,F(T)) = 0$, for each $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d\big(x_n,F(T)\big) < \frac{\epsilon}{M+1}$$

for all $n > n_1$. Thus, there exists $p_1 \in F(T)$ such that

$$d(x_n,p_1) < \frac{\epsilon}{M+1} \quad \textit{for all } n > n_1$$

and we obtain that

$$
\begin{aligned}
d(x_{n+m},x_n) &\le d(x_{n+m},p_1) + d(p_1,x_n) \\
&\le Md(x_n,p_1) + d(x_n,p_1)
\end{aligned}
$$

$$
\begin{aligned}
&= (M+1)d(x_n,p_1) \\
&< (M+1)\frac{\epsilon}{M+1} = \epsilon
\end{aligned}
$$

for all $m,n > n_1$. Therefore, $\{x_n\}$ is a Cauchy sequence in K. Since the set K is complete, the sequence $\{x_n\}$ must be convergence to a point in K. Let $\lim_{n\to\infty} x_n = p \in K$. Here after, we show that p is a fixed point. By $\lim_{n\to\infty} x_n = p$, for all $\epsilon_1 > 0$, there exists $n_2 \in \mathbb{N}$ such that

$$
d(x_n,p) < \frac{\epsilon_1}{2(2+u_1)} \tag{3.4}
$$

for all $n > n_2$. From (3.3), for each $\epsilon_1 > 0$, there exists $n_3 \in \mathbb{N}$ such that

$$
d\bigl(x_n,F(T)\bigr) < \frac{\epsilon_1}{2(4+3u_1)}
$$

for all $n > n_3$. In particular, $\inf\{d(x_{n_3},p) : p \in F(T)\} < \frac{\epsilon_1}{2(4+3u_1)}$. Thus, there must exist $p^\star \in F(T)$ such that

$$
d\bigl(x_{n_3},p^\star\bigr) < \frac{\epsilon_1}{2(4+3u_1)} \quad \text{for all } n > n_3. \tag{3.5}
$$

From (3.4) and (3.5),

$$
\begin{aligned}
d(Tp,p) &\le d\bigl(Tp,p^\star\bigr) + d\bigl(p^\star, Tx_{n_3}\bigr) + d\bigl(Tx_{n_3},p^\star\bigr) + d\bigl(p^\star,x_{n_3}\bigr) + d(x_{n_3},p) \\
&\le d\bigl(Tp,p^\star\bigr) + 2d\bigl(Tx_{n_3},p^\star\bigr) + d\bigl(x_{n_3},p^\star\bigr) + d(x_{n_3},p) \\
&\le (1+u_1)d\bigl(p,p^\star\bigr) + 2(1+u_1)d\bigl(x_{n_3},p^\star\bigr) + d\bigl(x_{n_3},p^\star\bigr) + d(x_{n_3},p) \\
&\le (1+u_1)d(p,x_{n_3}) + (1+u_1)d\bigl(x_{n_3},p^\star\bigr) + 2(1+u_1)d\bigl(x_{n_3},p^\star\bigr) \\
&\quad + d\bigl(x_{n_3},p^\star\bigr) + d(x_{n_3},p) \\
&= (2+u_1)d(x_{n_3},p) + (4+3u_1)d\bigl(x_{n_3},p^\star\bigr) \\
&< (2+u_1)\frac{\epsilon_1}{2(2+u_1)} + (4+3u_1)\frac{\epsilon_1}{2(4+3u_1)} = \epsilon_1.
\end{aligned}
$$

Since ϵ_1 is arbitrary, so $d(Tp,p) = 0$, *i.e.*, $Tp = p$. Therefore, $p \in F(T)$. This completes the proof. □

Remark 2 Let the hypothesis of Theorem 1 be satisfied and $T : K \to K$ be an asymptotically nonexpansive or quasi-nonexpansive mapping. From Remark 1, the class of asymptotically quasi-nonexpansive mappings includes quasi-nonexpansive mappings and asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Now, we give the following corollaries which have been proved by Theorem 1.

Corollary 1 *Under the hypothesis of Theorem* 1, T *satisfies the following conditions*:

(1) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(2) *If the sequence* $\{z_n\}$ *in* K *satisfies* $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, *then*

$$\liminf_{n\to\infty} d\bigl(z_n, F(T)\bigr) = 0 \quad \textit{or} \quad \limsup_{n\to\infty} d\bigl(z_n, F(T)\bigr) = 0.$$

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof It follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (2),

$$\liminf_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0.$$

Therefore, the sequence $\{x_n\}$ must converge to a fixed point of T by Theorem 1. □

Corollary 2 *Under the hypothesis of Theorem* 1, T *satisfies the following conditions*:

(1) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(2) *There exists a function* $f : [0,\infty) \to [0,\infty)$ *which is right-continuous at* 0, $f(0) = 0$ *and* $f(r) > 0$ *for all* $r > 0$ *such that*

$$d(x, Tx) \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr) \quad \textit{for all } x \in K,$$

where $d(x, F(T)) = \inf_{z\in F(T)} d(x, z)$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof It follows from the hypothesis that

$$\lim_{n\to\infty} f(d\bigl(x_n, F(T)\bigr) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is,

$$\lim_{n\to\infty} f(d\bigl(x_n, F(T)\bigr) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is right-continuous at 0 and $f(0) = 0$, therefore we have

$$\lim_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0.$$

Thus, $\liminf_{n\to\infty} d(x_n, F(T)) = \limsup_{n\to\infty} d(x_n, F(T)) = 0$. By Theorem 1, the sequence $\{x_n\}$ converges strongly to q, a fixed point of T. This completes the proof. □

Finally, we give the following theorem which has a different hypothesis from Theorem 1.

Theorem 2 *Let* K *be a nonempty closed convex subset of a complete* CAT(0) *space* X, $T : K \to K$ *be an asymptotically quasi-nonexpansive mapping with* $F(T) \neq \emptyset$ *and* $\{u_n\}$ *be a nonnegative real sequence with* $\sum_{n=1}^{\infty} u_n < \infty$. *Suppose that* $\{x_n\}$ *is defined by the iteration process* (1.7). *If* T *is semi-compact and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof From the hypothesis, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, since T is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. Hence,

$$d(p, Tp) = \lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

Thus, $p \in F(T)$. By (3.2),

$$\begin{aligned} d(x_{n+1}, p) &\le (1 + u_n)^2 d(x_n, p) \\ &= \left(1 + 2u_n + u_n^2\right) d(x_n, p). \end{aligned}$$

Since $\sum_{n=1}^{\infty} u_n < \infty$, we have $\sum_{n=1}^{\infty}(2u_n + u_n^2) < \infty$. By Lemma 2, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_k} \to p \in F(T)$ gives that $x_n \to p \in F(T)$. This completes the proof. □

4 Conclusions

The class of quasi-nonexpansive mappings and asymptotically nonexpansive mappings includes nonexpansive mappings, where as the class of asymptotically quasi-nonexpansive mappings is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings. Then these results presented in this paper extend and generalize some works for a CAT(0) space in the literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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References

1. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings. Dekker, New York (1984)
2. Bridson, M, Haefliger, A: Metric Spaces of Non-Positive Curvature. Springer, Berlin (1999)
3. Brown, KS: Buildings. Springer, New York (1989)
4. Burago, D, Burago, Y, Ivanov, S: A Course in Metric Geometry. Graduate Studies in Math., vol. 33. Am. Math. Soc., Providence (2001)
5. Gromov, M: Metric Structures for Riemannian and Non-Riemannian Spaces. Progress in Math., vol. 152. Birkhäuser, Boston (1999)
6. Bartolini, I, Ciaccia, P, Patella, M: String Matching with Metric Trees Using an Approximate Distance. SPIR Lecture Notes in Computer Science, vol. 2476. Springer, Berlin (1999)
7. Semple, C: Phylogenetics. Oxford Lecture Series in Mathematics and Its Application. Oxford University Press, Oxford (2003)
8. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Math. Anal., Malaga/Seville, 2002/2003. Colecc. Abierta, vol. 64, pp. 195-225. Univ. Seville Secr. Publ., Seville (2003)
9. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theo. Appl., pp. 113-142. Yokohama Publ., Yokohama (2004)
10. Dhompongsa, S, Kirk, WA, Sims, B: Fixed point of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
11. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear and Convex Anal. **8**, 35-45 (2007)
12. Dhompongsa, S, Panyanak, B: On △-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572-2579 (2008)
13. Razani, A, Salahifard, H: Invariant approximation for CAT(0) spaces. Nonlinear Anal. **72**, 2421-2425 (2010)
14. Khan, SH, Abbas, M: Strong and △-convergence of some iterative schemes in CAT(0) spaces. Comput. Math. Appl. **61**, 109-116 (2011)
15. Agarwal, RP, O'Regan, D, Sahu, DR: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex. Anal. **8**(1), 61-79 (2007)

16. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Austral. Math. Soc. **43**, 153-159 (1991)
17. Tan, KK, Xu, HK: Fixed point iteration process for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. **122**(3), 733-739 (1994)
18. Diaz, JB, Metcalf, FT: On the structure of the set of subsequential limit points of successive approximations. Bull. Amer. Math. Soc. **73**, 516-519 (1967)
19. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. **35**(1), 171-174 (1972)
20. Liu, Q: Iterative sequences for asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. **259**, 1-7 (2001)
21. Fukhar-ud-din, H, Khan, SH: Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications. J. Math. Anal. Appl. **328**, 821-829 (2007)
22. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0) spaces. Nonlinear Anal. **74**, 783-791 (2011)
23. Beg, I, Abbas, M: An iterative process for a family of asymptotically quasi-nonexpansive mappings in CAT(0) spaces. Novi Sad J. Math. **41**(2), 149-157 (2011)
24. Kirk, WA: A fixed point theorem in CAT(0) spaces and *R*-trees. Fixed Point Theory Appl. **4**, 309-316 (2004)
25. Bruhat, F, Tits, J: Groupes r éductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math. **41**, 5-251 (1972)
26. Zhou, H, Agarwal, RP, Cho, YJ, Kim, YS: Nonexpansive mappings and iterative methods in uniformly convex Banach spaces. Georgian Math. J. **9**, 591-600 (2002)

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